By (2), $\rho^2\dot{\theta} = A$, so that (1) gives

$$\ddot{\rho} - A^2\rho^{-3} = -\mu\rho^{-2},$$

or

$$\dot{\rho} = \sqrt{-A^2\rho^{-2} + 2\mu\rho^{-1} + B}.$$

The speed is

$$\sqrt{\dot{\rho}^2 + \rho^2\dot{\theta}^2} = \sqrt{2\mu\rho^{-1} + B}.$$

Hence,

$$B = v^2 - 2\mu a^{-1}.$$

Hence,

$$\frac{d\rho}{d\theta} = \frac{\dot{\rho}}{\dot{\theta}} = \rho^2 A^{-1}\sqrt{v^2 - 2\mu a^{-1} + 2\mu\rho^{-1} - A^2\rho^{-2}}$$

and therefore,

$$\theta = -\int_{a^{-1}}^{\rho^{-1}} A^{-1}\sqrt{v^2 - 2\mu a^{-1} + 2\mu z - A^2 z^2}\, dz;$$

that is,

$$\rho^{-1} = \mu A^{-2} - (\mu A^{-2} - a^{-1})\cos\theta + \sqrt{v^2 A^{-2} - a^{-2}}\cdot\sin\theta,$$

which is the equation of the trajectory.

Writing this in the form,

$$\mu^2 a^2(1-\cos\theta)^2 + \{2\mu a(1-\cos\theta)(\cos\theta - a\rho^{-1}) - a^2v^2\sin^2\theta\}A^2 + \{(\cos\theta - a\rho^{-1})^2 + \sin^2\theta\}A^4 = 0,$$

and applying the condition for equal roots in A^2, we get for the envelope

$$\frac{4\mu a^2v^2}{4\mu^2 - a^2v^4}\frac{1}{\rho} = 1 - \frac{2\mu - av^2}{2\mu + av^2}\cos\theta,$$

the equation of an ellipse.

349 (Mechanics). Proposed by S. A. COREY, Albia, Iowa.

A 9-pound weight is attached to a string which passes over a smooth fixed pulley. The other end of the string is fastened to and supports a smooth pulley P_1 of weight 1 pound over which passes a second string, one end attached to a 3-pound weight and the other end attached to and supporting another smooth pulley P_2 of weight 1 pound. Over the pulley P_2 passes a third string supporting weights 2 pounds and $3\frac{1}{3}$ pounds.

If the system is acted upon by gravity alone show that the acceleration of the 9-pound weight, $3\frac{1}{3}$-pound weight, and pulley P_2 are 0, $\frac{1}{2}g$, and $\frac{1}{3}g$, respectively.

Determine the motion of the weights when pulleys are not smooth, that is, when friction is present.

II. Solution by William Hoover, Columbus, Ohio.

Calling the fixed pulley P and taking it as the origin of coördinates, we have for the x-coördinate of m_1, x_1; of P_1, $l_1 - x_1$; of m_3, $x_2 + l_1 - x_1$; of P_2, $l_1 + l_2 - (x_1 + x_2)$; of m_4, $x_3 + l_1 + l_2 - (x_1 + x_2)$; of m_5, $l_1 + l_2 + l_3 - (x_1 + x_2 + x_3)$, where m_1, m_3, m_4 and m_5 are the masses of the 9-, 3-, 2-, and $3\frac{1}{3}$-pound weights, respectively.

Under the hypothesis that there is no friction, the equation of motion of system is

$$\begin{aligned} T = \tfrac{1}{2}m_1\dot{x}_1^2 + \tfrac{1}{2}P_1\dot{x}_1^2 + \tfrac{1}{2}m_3(\dot{x}_2 - \dot{x}_1)^2 + \tfrac{1}{2}(\dot{x}_1 + \dot{x}_2)^2 + \tfrac{1}{2}m_4(\dot{x}_3 - \dot{x}_1 - \dot{x}_2)^2 + \tfrac{1}{2}m_5(\dot{x}_1 + \dot{x}_2 \\ + \dot{x}_3)^2 = m_1gx_1 + P_1g(l_1 - x_1) + m_3g(x_2 + l_1 - x_1) + P_2g\{l_1 + l_2 - (x_1 + x_2)\} \\ + m_4g\{l_1 + l_2 + x_3 - (\dot{x}_1 + x_2)\} + C = V. \end{aligned} \qquad (i)$$

Using Lagrange's equations of type

$$\frac{d}{dt}\frac{dT}{d\dot{x}} - \frac{dT}{dx} = \frac{dV}{dx} \qquad (ii)$$

there are

$$\begin{aligned} &(a) \quad \tfrac{58}{3}\ddot{x}_1 + \tfrac{10}{3}\ddot{x}_2 + \tfrac{4}{3}\ddot{x}_3 = g(m_1 - P_1 - m_3 - P_2 - m_4 - m_5) = -\tfrac{4}{3}g, \\ &(b) \quad \tfrac{10}{3}\ddot{x}_1 + \tfrac{28}{3}\ddot{x}_2 + \tfrac{4}{3}\ddot{x}_3 = g(m_3 - P_2 - m_4 - m_5) = -\tfrac{10}{3}g, \\ &(c) \quad \tfrac{4}{3}\ddot{x}_1 + \tfrac{4}{3}\ddot{x}_2 + \tfrac{16}{3}\ddot{x}_3 = g(m_4 - m_5) = -\tfrac{4}{3}g, \end{aligned}$$

giving $\ddot{x}_1 = 0$, $\ddot{x}_2 = -\frac{g}{3}$, $x_3 = -\frac{g}{6}$.

[In the previously printed solution nothing is given to show *how* the *literal* terms on right of (*a*), (*b*), (*c*) arise.]

In case of friction of the movable pulleys, account generally would have to be taken of the *vis viva* of rotary motion of P_1 and P_2, there being, then, two additional terms in the energy equation $T = V$, namely,

$$\frac{k_1^2}{2} P_1\dot{\varphi}_1^2 \quad \text{and} \quad \frac{k_2^2}{2} P_2\dot{\varphi}_2^2,$$

k_1 and k_2 being the radii of gyration of the pulleys P_1, P_2, supposed to be cylinders of radii, say, r_1, r_2, and so $k_1^2 = r_1^2/2$, $k_2^2 = r_2^2/2$, and $\dot{\varphi}_1$, $\dot{\varphi}_2$, the angular velocities of P_1, P_2.

There were not stated any dimensions of P_1, P_2; permitting us to take r_1 and r_2 indefinitely small. This would, though, be a special condition.

In taking r_1, r_2 of finite value, the energy equation would be

$$T = \tfrac{1}{2}(m_1 + P_1 + m_3 + P_2 + m_4 + m_5)\dot{x}_1^2 + \text{etc.} + (-m_3 + P_2 + m_4 + m_5)\dot{x}_1\dot{x}_2 + \text{etc.}$$
$$+ \tfrac{1}{2}k_1^2\dot{\varphi}_1^2 + \tfrac{1}{2}k_2^2\dot{\varphi}_2^2 = m_1gx_1 + \text{etc.} + C = V. \quad (d)$$

Regardless of signs, $r_1\,\dot{\varphi}_1 = \dot{x}_1$, $r_2\,\dot{\varphi}_2 = \dot{x}_2$; and substituting $\dot{\varphi}_1$, $\dot{\varphi}_2$ from these equations in (*d*), we have an equation differing in form from (*i*), but to which (*ii*) may be applied as before, and generally with different values for $\ddot{x}_1$ etc.

So far the fixed pulley has been thought of as smooth. If this pulley be regarded as rough, the energy equation would still further be modified so as to take account of work done against friction, or of energy due to its rotation and some assigned mass

350 (Mechanics). Proposed by J. B. REYNOLDS, Lehigh University.

If an elastic tube filled with liquid under pressure doubles in length in what ratio will the radius be increased?

SOLUTION BY THE PROPOSER.

Let the initial length and radius of the tube be l_0 and r_0, the final lengths l and r and the coefficient of elasticity of a strip of the material one unit wide be λ.

If P is the pressure per square unit we have, since the length is doubled,

$$P\pi r^2 = 2\pi r \frac{\lambda}{l_0} l_0; \qquad \text{whence} \qquad P = \frac{2\lambda}{r}. \tag{1}$$

Again, if T is the peripheral tension per unit length, in order to balance the internal pressure

$$T = \lim_{\Delta\theta \doteq 0} \frac{Pr\Delta\theta}{2\sin\Delta\theta/2} = Pr. \tag{2}$$

Also,

$$T = \frac{\lambda}{2\pi r_0}(2\pi r - 2\pi r_0) = \frac{\lambda}{r_0}(r - r_0). \tag{3}$$

By (1), (2), and (3) we get,

$$Pr = 2\lambda = \frac{\lambda}{r_0}(r - r_0);$$

whence $r = 3r_0$.

356 (Mechanics). Proposed by FLORENCE P. LEWIS, Goucher College, Baltimore, Md.

A ray of light enters a prism having vertex angle α. If the angle between the incoming and outgoing directions is defined as the angle of deviation, at what angle must the ray enter the prism in order that the angle of deviation may be a minimum?

SOLUTION BY J. B. REYNOLDS, Lehigh University.

We have

$$\sin\varphi = \epsilon\sin x, \qquad \sin\psi = \epsilon\sin(\alpha - x),$$